Math Majors of America Tournament for High Schools
Sample Contest

1 Individual Sample

Individual Round A **10 minutes**

1. Compute the area of the triangle with vertices at the coordinates (1337, 1337), (1337, 1344), and (1349, 1337).

2. Three points are selected randomly (and uniformly) along the circumference of a circle with radius 1. Compute the probability that they are the vertices of an acute triangle.

Individual Round B **10 minutes**

3. Compute the largest real x such that
$$x - 3\sqrt{x-3} = \frac{1}{\sqrt{x-3}}.$$

4. The sum $\frac{7}{12} + \frac{9360}{d}$ is an integer. Compute the smallest positive integral value for d.

Individual Round C **10 minutes**

5. Compute the minimum value of $f(x) = \sqrt{x^2 - 8x + 21}$.

6. Compute the sum of the solutions to the equation $\sin(x) + 8\cos^2(x)\sin(x) + 3\sqrt{3}\cos(x) = 0$ on the interval $0 < x < 2\pi$.

2 Individual Solutions

1. The triangle is right and has side lengths of 12 and 7. So $A = \frac{1}{2}(12)(7) = 42$.

2. We calculate the complementary probability – that the triangle is obtuse. Note that the probability of a right triangle is zero. The triangle will be obtuse if and only if all three points lie on the same side of some diameter. Let the random variable X_k be associated with the point on the circle P_k, for $k = 1, 2, 3$, and define it as 1 if the other two points are within π clockwise from it and 0 otherwise. We are looking for
$P(X_1 + X_2 + X_3 = 1) = E(X_1 + X_2 + X_3)$ since at most one of these variables can be 1. Because the points are chosen uniformly, their probability distributions are identical. So the above equals $3E(X_1) = \frac{3}{4}$ because the probability that each of the other two points lie in the semicircle starting at X_1 is $\frac{1}{2} \cdot \frac{1}{2}$. Our desired probability is then $1 - \frac{3}{4}$.

3. Add and subtract 3 to the left hand side:
$(x-3) - 3\sqrt{x-3} + 3 = \frac{1}{\sqrt{x-3}} \Rightarrow (x-3)^{\frac{3}{2}} - 3(x-3)^{\frac{2}{2}} + 3(x-3)^{\frac{1}{2}} - 1 = 0 \Rightarrow ((x-3)^{\frac{1}{2}} - 1)^3 = 0 \Rightarrow (x-3) = 1$.
Therefore $x = 4$ is the only real solution.

4. Write the second fraction as $\frac{c \cdot m}{c \cdot n}$ where the greatest common divisor of m and n is 1. If $\frac{7}{12} + \frac{m}{n}$ is to be some integer k, then $7n + 12m = 12nk \Rightarrow 12m = 12nk - 7n \Rightarrow m = nk - 7 \cdot \frac{n}{12}$. Therefore 12 divides n. We also have that $7n = 12nk - 12m \Rightarrow 7 = 12k - 12\frac{m}{n}$. But since m and n are relatively prime, we have that n divides 12. So $n = 12$. Therefore our goal is to have that when the fraction is reduced the denominator is 12. We need to find the smallest c so that the numerator 9360 has no factors of 12. Dividing 9360 by 12 two times shows that $9360 = 12^2 \cdot 65$. So $c = 12^2$ for the smallest value of d, and $d = 12^2 \cdot 12 = 1728$. We must check that this is indeed an integer, and it is because $\frac{7}{12} + \frac{65}{12} = \frac{72}{12} = 6$. So $d = 1728$.

5. Note that completing the square inside the radical yields $f(x) = \sqrt{(x-4)^2 + 5}$. Notice that $(x-4)^2$ has a minimum value at $x = 4$, leaving 0. Hence, the minimum value of $f(x)$ is $\sqrt{5}$.

6. Our motivation for the solutions comes from factoring the first two terms and trying to write the summation as (close to) symmetric with both $\sin^3(x)$ and $\cos^3(x)$ terms. Making use of the identity $\sin^2(x) + \cos^2(x) = 1$ we see that

$$\begin{aligned}
\sin(x) + 8\cos^2(x)\sin(x) + 3\sqrt{3}\cos(x) &= \sin(x)(1 + 8\cos^2(x)) + 3\sqrt{3}\cos(x) \\
&= \sin(x)((\sin^2(x) + \cos^2(x)) + 8\cos^2(x)) + 3\sqrt{3}\cos(x) \cdot 1 \\
&= \sin(x)(\sin^2(x) + 9\cos^2(x)) + 3\sqrt{3}\cos(x)(\cos^2(x) + \sin^2(x)) \\
&= \sin^3(x) + 9\sin(x)\cos^2(x) + 3\sqrt{3}\cos^3(x) + 3\sqrt{3}\sin^2(x)\cos(x) \\
&= \sin^3(x) + 3\sqrt{3}\sin^2(x)\cos(x) + 9\sin(x)\cos^2(x) + 3\sqrt{3}\cos^3(x) \\
&= (\sin(x) + \sqrt{3}\cos(x))^3
\end{aligned}$$

This is only zero when $\sin(x) + \sqrt{3}\cos(x) = 0 \Rightarrow \tan(x) = -\sqrt{3}$, which happens when $x = \frac{2\pi}{3}$ or $\frac{5\pi}{3}$. The sum is $\frac{7\pi}{3}$.

Note that at the actual competition there will be more than 3 rounds.

3 Mathathon Sample Questions

Mathathon Round 1 – 2 pts each

1. What is the largest distance between any two points on a regular hexagon with a side length of one?

2. For how many integers $n \geq 1$ is $\dfrac{10^n - 1}{9}$ the square of an integer?

Mathathon Round 3 – 4 pts each

3. A vector in 3D space that in standard position in the first octant makes an angle of $\frac{\pi}{3}$ with the x axis and $\frac{\pi}{4}$ with the y axis. What angle does it make with the z axis?

4. Compute $\sqrt{2012^2 + 2012^2 \cdot 2013^2 + 2013^2} - 2012^2$.

5. Round $\log_2 \left(\sum_{k=0}^{32} \binom{32}{k} \cdot 3^k \cdot 5^k \right)$ to the nearest integer.

Mathathon Round 6 – 8 pts each

6. Let P be a point inside a ball. Consider three mutually perpendicular planes through P. These planes intersect the ball along three disks. If the radius of the ball is 2 and $\frac{1}{2}$ is the distance between the center of the ball and P, compute the sum of the areas of the three disks of intersection.

7. Find the sum of the absolute values of the real roots of the equation $x^4 - 4x - 1 = 0$.

Mathathon Round 8 – 10 pts each

8. The numbers $1, 2, 3, ..., 2013$ are written on a board. A student erases three numbers a, b, c and instead writes the number $\frac{1}{2}(a + b + c) \left((a - b)^2 + (b - c)^2 + (c - a)^2\right)$ She repeats this process until there is only one number left on the board. List all possible values of the remainder when the last number is divided by 3.

9. How many ordered triples of integers (a, b, c), where $1 \leq a, b, c \leq 10$, are such that for every natural number n, the equation $(a + n)x^2 + (b + 2n)x + c + n = 0$ has at least one real root?

4 Mathathon Solutions

1.Note that the vertices of a regular hexagon all lie on the circumcircle of the hexagon. Then the furthest distance from one vertex is to another point on the circle, or the opposite vertex. Since a radius of the circle is the length of a side, the total distance is 2.

2. Since $\frac{10^n - 1}{10 - 1}$ is the sum of a geometric series with first term 1 and common ratio 10, the numbers of this form are exactly the numbers 1, 11, 111, etc. Clearly 1 is a square (corresponding to $n = 1$). But modulo 8, this sequence is 1, 3, 7, 7, 7, ..., and it will remaind a constant of 7 because every power of 10 that is added from then on is divisible by 8. On the other hand, the squares modulo 8 are 0, 1, and 4. So there are no other perfect squares mod 8, which means there are no other perfect squares in the original sequence.

3. Note that for an axis e_i and a vector $\vec{x}$, where ω_i is the angle between them, then $\cos(\omega_i) = \frac{1}{|\vec{x}|}(0, 0, 0, \ldots, 1, \ldots 0) \cdot \vec{x}$. Summing the squares of these must give 1. So $\cos^2(\frac{\pi}{3}) + \cos^2(\frac{\pi}{4}) + \cos^2(\theta) = 1 \Rightarrow \theta = \frac{\pi}{3}$.

4. Set $a = 2012$ so the radicand is $a^2 + a^2 \cdot (a+1)^2 + (a+1)^2 = (a^2 + a + 1)^2$. So we're looking for $2012^2 - 2012^2 + 2013 = 2013$.

5. We can write the summation as $\binom{32}{k} \cdot 15^k$, which counts, for every value of k, the number of ways to choose k balls out of our starting 32 balls in one bucket and place them into 15 other buckets. Summing over all k counts the total number of ways to distribute the 32 balls among 16 buckets, which is equal to $16^{32} = 2^{128}$. (Note that this expression is also the expansion of $(1 + 15)^{32}$).

6. Let d_1, d_2, d_3 be coordinates of P in some rectangular coordinate system with the origin at the center of the ball and whose coordinates are parallel to the three mutually orthogonal planes through P. Then $d^2 = d_1^2 + d_2^2 + d_3^2$. Moreover, the radii of the disks of intersection are $\sqrt{R^2 - d_1^2}$, $\sqrt{R^2 - d_2^2}$, and $\sqrt{R^2 - d_3^2}$. Hence the sum of their areas are $\pi(2^2 - d_1^2 + 2^2 - d_2^2 + 2^2 - d_3^2) = \pi(12 - \frac{1}{4}) = \frac{47\pi}{4}$.
Remark: The result does not depend on the choice of mutually perpendicular planes through P.

7. $\sqrt{4\sqrt{2} - 2}$Note that $x^4 - 4x - 1 = (x^2 + 1)^2 - 2(x+1)^2 = 0 \Rightarrow x^2 + 1 = \sqrt{2}(x + 1)$. Hence, $x = \frac{1 \pm \sqrt{2\sqrt{2} - 1}}{\sqrt{2}}$. By Descarte's rule of signs, these are the only two real roots.

8. Observe that

$$\frac{1}{2}(a + b + c)\left((a - b)^2 + (b - c)^2 + (c - a)^2\right) = a^3 + b^3 + c^3 - 3abc \equiv a + b + c \pmod 3$$

So the sum of the numbers on the board is constant viewed modulo 3. The initial sum is $\frac{2013 \cdot 2014}{2} \equiv 0 \pmod 3$, so the remaining number in the end must be divisible by 3.

9. In order for the equation $(a + n)x^2 + (b + 2n)x + (c + n) = 0$ to have a real root, we need that $(b + 2n)^2 - 4(a + n)(c + n) \geq 0$, which is equivalent with $b^2 - 4ac + 4n(b - a - c) \geq 0$. It is enough that $b - a - c \geq 0$ because $b^2 - 4ac \geq 0$ follows from the fact that $b \geq a + c$ because $b^2 \geq (a + c)^2 \geq 4ac$. The case $b^2 - 4ac \geq 0$ and $b < a + c$ is not possible because when we let n become large enough the number $4n(b - a - c)$ is very small (i.e. a very negative number), whilst $b^2 - 4ac$ is a finite positive number and in this case we would have $b^2 - 4ac + 4n(b - a - c) < 0$ for all n large enough, a contradiction. Therefore it is enought to find those triples so that $b \geq a + c$.
For $b = 1$ there is no pair (a, c) so that $b \geq a + c$. If $b = 2$ the only pair is $(1, 1)$. If $b = 3$ then $3 \geq 1 + 2 = 2 + 1$ and $3 > 1 + 1$. In general, if $x_1, x_2, ..., x_m$ are natural numbers and $m \leq n$ then the equation $x_1 + x_2 + ... + x_m = n$ has $\binom{n-1}{m-1}$ solutions. In our case $m = 2$. So we find all those cases when $b = a + c, b - 1 = a + c, b - 2 = a + c$ etc. Also $\binom{n-1}{1} = n - 1$, therefore we find that the desired number is:

$$\sum_{i=2}^{10} \frac{i(i-1)}{2} = 165$$

Note that at the actual competition, each round will have exactly 3 questions.

5 Relay Samples

The first two sets are 3 person relays.

1-1. Mitchell and Sitharthan decide to meet at the Reitz for lunch at some time between 10 a.m. and 11 a.m. They both forgot what time precisely they decided on meeting and randomly arrive (uniformly) during that interval. Mitchell will wait for 10 minutes after he arrives, and Sitharthan will wait for 10 minutes also. Let the probability that meet each other be $\frac{m}{n}$ for relatively prime positive integers m and n. What is $m+n$?

1-2. Let $T = TNYWR$. The smallest integral value of a strictly greater than 31 such that the equation $6Tx + a \cdot y = 1$ has solutions $(x,\, y)$ in the integers is Q. Compute $Q - 27$.

1-3. Let $T = TNYWR$. Three circles with radii 3, 4, and $(T-3)$ are all mutually externally tangent. What is the area of the triangle with vertices at their centers?

2-1. Find the sum of the solutions for x in the following equation where $0 < x \leq 2\pi$

$$\cot(x) + \tan(x) = 4$$

2-2. Let $T = TNYWR$. Given $z = \cos(\frac{2T}{9}) + i\sin(\frac{2T}{9})$, compute $\frac{2}{1+z}$ and express in the form $a + ib$.

2-3. Let $T = TNYWR$. An infinite number of pitches are thrown to Tony during batting practice. The probability that Tony hits exactly n of these pitches is P_n (for $n = 0, 1, 2, \ldots$), where $P_{n+1} = \frac{1}{3}P_n$ for $n \geq 0$. What is the probability that Tony hits exactly $|T|^2$ pitches?

This set is a 6 person relay, and 5-6 is the final answer.

3-1. How many 3-digit (base-10) positive integers are odd and do not contain the digit 4? (Pass answer to 3-2)

3-2. Let $T = TNYWR$. Joanna, the party animal, invited $\lfloor \frac{T}{60} \rfloor$ of her friends to go to a show about growing old with cats (so, in total, $\lfloor \frac{T}{60} \rfloor + 1$ people are going). If they randomly sit in a row, then the probability that Joanna is in the middle is $\frac{m}{n}$ for relatively prime positive integers m and n. Find $m + n$. (Pass answer to 3-3)

3-3. Let $T = TNYWR$. How many (base-10) positive integers greater than $50 \cdot T$ can be formed using only the digits in the set $\{1, 2, 3, 4, 5\}$, where the digits in any given number are all distinct? (Pass answer to 3-6)

3-4. Find the value of q in the following system of equations

$$(\log_3 p)^2 = \log_3 p^2 \text{ and } \log_3(p + q) = \log_3 p + \log_3 q$$

(Pass answer to 3-5)

3-5. Let $T = TNYWR$. Find the area of the triangle in the complex plane with vertices at the complex cube roots of $\lfloor T \rfloor$. (Pass answer to 3-6)

3-6. Let the larger of the two numbers you receive be A and the smaller be B and let $Z = \lfloor \frac{A}{40} \rfloor$ and $T = \frac{(4B)^2}{9}$. How many ways can T balls be chosen from a bag with Z balls of distinct colors (no balls are identical to each other all are distinct) with replacement if order does not matter?

6 Relay Solutions

1-1: Draw a square with side length 60 (for 60 minutes), then go up 10 units on the x and y axis on opposing corners. Shade in the region contained by these points, this shaded region is the probability that they will eat lunch together. To calculate that area find the area of the two isoceles triangles that are left over $(50 * 50)$ and now subtract by the toal area $3600 - 2500 = 1100$. The probability is $\frac{11}{36}$ and $m + n$ equals $\boxed{47}$.
1-2: In order for the condition to be true $6T$ and a must be relatively prime $6T$ factorizes into $2 * 3 * 47$ for a to be relatively prime it must be a multiple of 5, since 4 is a multiple of 2 it would not be relatively prime. The smallest multiple of 5 greater than 30 is 35. So $Q - 27 = \boxed{8}$.
1-3: The legs of the triangle are 7, T, and $T + 1$. We use Heron's formula to get the area. The semiperimeter is $T + 4$. So $A = \sqrt{(T+4)(T-3)(4)(3)}$. Since $T = 8$, $A = \sqrt{12 \cdot 5 \cdot 12} = \boxed{12\sqrt{5}}$.

2-1: The equation simlifies to $\sin(x)\cos(x) = \frac{1}{4}$, which further simplifies to $sin(2x) = \frac{1}{2}$ Note that due to the $2x$ you have to account for 4 solutions which are $\frac{\pi}{12}$, $\frac{5\pi}{12}$, $\frac{13\pi}{12}$, $\frac{17\pi}{12}$. The solutions add up to $\boxed{3\pi}$.
2-2: Let $x = \frac{T}{9}$, thus $1 + z = 1 + \cos(2x) + i\sin(2x)$ which equals $2\cos^2(x) + i\sin(2x)$ changing everything into $\tan(x)$ gives $\frac{2}{1+\tan^2(x)} + \frac{2i\tan(x)}{1+\tan^2(x)}$ adding fractions gives $\frac{2[1+i\tan(x)]}{1+\tan^2(x)}$ which simplifies to $\frac{2}{1-i\tan(x)}$ plugging this into the original equation gives $1 - i\tan(\frac{T}{9})$. Plugging in $T = 3\pi$ gives $\boxed{1 - i\sqrt{3}}$.
2-3: The probabilities follow the pattern P_0, $\frac{1}{3}P_0$, $\frac{1}{9}P_0$, ... The sum of this series has to equal 1 because it contains all the possible probabilities. Factoring out the P_0 and doing a geometric series on the sum yields $\frac{3}{2}$ thus $\frac{3}{2} \cdot P_0 = 1$ and P_0 equals 2/3. Thus P_4 is $\boxed{\dfrac{2}{243}}$.

3-1: For a number to be odd it must end in 1,3,5,7, or 9. Thus there is 5 possiblities for the last digit 9 for the second and 8 for the first (cannot be 0). $9 * 8 * 5 = \boxed{360}$.
3-2: There are 7! ways to arrange 7 people in a row with no restrictions. With the restriction that Joanna has to be in the middle there are 6 other spots and 6! ways to arrange those spots. The probability is $\frac{6!}{7!}$ whch reduces to $\frac{1}{7}$ $1 + 7 = \boxed{8}$.
3-3: 50T=400. For 3 digit numbers greater than 400 the hundreds digit can be 4 or 5. The tens digit has 4 possiblities and the ones digit has 3. Thus for 3 digit numbers the number of possiblities is $2 * 4 * 3$. For 4 digit and 5 digit numbers there are 5! numbers for each. The sum of possbilities is equal to $4! + 2 * 5! = \boxed{264}$. 3-4: Let $x = \log_3 p$. The first equation then reduces to x=2, which makes $p = 9$. Plugging $p = 9$ into the second equation gives $\log_3(9 + q) = \log_3 9 + \log_3 q$ and using log rules that simplifies to $\log_3(9 + q) = \log_3 9q$ further simplifying gives $9 + q = 9q$ solving for q gives $q = \boxed{9/8}$
3-5: To solve for the cube roots of 1, divide the unit circle into 3 sectors with the first point at $(1, 0)$, the other two points will be at $(\frac{-1}{2}, \frac{\sqrt{3}}{2})$ and $(\frac{-1}{2}, \frac{-\sqrt{3}}{2})$ these points are already on the complex plane so you can just calculate the area of the triangle formed by the points using a variety of methods. The calculated area is $\boxed{\dfrac{3\sqrt{3}}{4}}$
3-6: The generalized solution to this problem is $\binom{n+r-1}{r}$. The problem is the equivalent of asking how many ways can you arrange r bugs on n leaves. To solve consider $|$ as the border of a leaf and o as a bug so if we had 4 leaves and 2 bugs one combination would be $|o||o||$ where the 1st and 3rd levaes have bugs on them. To find the number of combinations we need to see how many different way we can arrange the inside because the outside must always remain bars (the border of the leaves). So on the inside we have $n + r - 1$ items which can be arranged in $n + r - 1!$ ways but since the bugs and leaves are indistinguishable you have to divide this by $r!$ and $n - 1!$ giving us the generalized solution of $\binom{n+r-1}{r}$. Plugging in the values given yields a final answer of $\boxed{56}$.

Note that on the day of the competition there will be a different number of total relays and types.

7 Tiebreaker Sample

Tiebreaker Round One **GaMaCo 2013**

1. When the least common multiple of two positive integers is divided by their greatest common divisor, the result is 21. One of the integers is 161. Compute the smallest possible value of the other integer.

Solution: Let the other number be n. Then $\frac{lcm(161, n)}{gcd(161, n)} = 21$, and we also know that $lcm(161, n) \cdot gcd(161, n) = 161n$. Dividing this equation by the previous, we obtain $(gcd(161, n))^2 = \frac{161n}{21} = \frac{23n}{3}$. Thus, $3(gcd(161, n))^2 = 23n$. As $gcd(3, 23) = 1$, we see that it must be the case that both $3 \mid n$ and $23 \mid n$, so the smallest possible value is $23 \cdot 3 = \boxed{69}$.

Note that on the day of the comeptition there will likely be more than one tiebreaker round of varying difficulties.